

UNCLASSIFIED

AD 400 398

*Reproduced
by the*

**ARMED SERVICES TECHNICAL INFORMATION AGENCY
ARLINGTON HALL STATION
ARLINGTON 12, VIRGINIA**



UNCLASSIFIED

NOTICE: When government or other drawings, specifications or other data are used for any purpose other than in connection with a definitely related government procurement operation, the U. S. Government thereby incurs no responsibility, nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use or sell any patented invention that may in any way be related thereto.

**APPLIED MATHEMATICS AND STATISTICS LABORATORIES
STANFORD UNIVERSITY
CALIFORNIA**

**GENERALIZED LEAST SQUARES ESTIMATORS FOR
RANDOMIZED FRACTIONAL REPLICATION DESIGNS**

**BY
S. ZACKS**

**TECHNICAL REPORT NO. 86
March 15, 1963**

**PREPARED UNDER CONTRACT Nonr-225(52)
(NR-342-022)
FOR
OFFICE OF NAVAL RESEARCH**



400 398

GENERALIZED LEAST SQUARES ESTIMATORS FOR
RANDOMIZED FRACTIONAL REPLICATION DESIGNS

by
S. Zacks

TECHNICAL REPORT NO. 86

March 15, 1963

PREPARED UNDER CONTRACT Nonr-225(52)

(NR-342-022)

FOR
OFFICE OF NAVAL RESEARCH



Reproduction in Whole or in Part is Permitted for
any Purpose of the United States Government

APPLIED MATHEMATICS AND STATISTICS LABORATORIES

STANFORD UNIVERSITY

STANFORD, CALIFORNIA

Generalized Least Squares Estimators
for Randomized Fractional Replication Designs

by
S. Zacks

1. Introduction

Fractional replication designs have become of great importance, especially for industrial experimentation. A missile whose operation is affected simultaneously by dozens of interacting factors would produce a full factorial experiment of impractical size. Indeed, if there are more than 20 factors which may affect the operation of a missile, and if we like to attain complete information on all the main effects and interactions of the controllable factors we would run the experiments over more than $2^{20} = 1,048,576$ treatment combinations. Fractional replication designs are planned to attain information about some of the main effects and interactions by a relatively small number of trials. If the operation of a missile can be controlled with some information on the main effects and some low order linear interactions, it might be sufficient to run only 32 or 64 trials at a time. These however, should be chosen from those possible in some optimal manner.

The problem of choosing a $1/2^{m-s}$ fractional replication and an appropriate estimator of the parameters characterizing the factorial model (main effects and interactions) has been studied by A. P. Dempster (1960, 1961), K. Takeuchi (1961), S. Ehrenfeld and S. Zacks (1961, 1962), S. Zacks (1962), B. V. Shah and O. Kempthorne (1962a,b). In all these studies the type of estimators considered is that which yields, under a randomized procedure with equal probabilities of choice, unbiased

estimates of a specified linear functional of a subvector of parameters, which lies in the range of the design matrix (the matrix of the corresponding normal equations).

In the present study statistical properties of the generalized least squares estimators, under randomized fractional replication designs, are studied. The term generalized least-squares estimators (denoted henceforth by g.l.s.e.) is used since the matrices of the normal equations corresponding to these designs are singular. The factorial models corresponding to the type of fractional replication designs studied in the present paper is presented in section 2. For this sake we start from the factorial model for a full factorial system. Then we present the required algebra, and the method of constructing the orthogonal fractional replications. The linear spaces of all g.l.s.e. associated with the various orthogonal fractional replication designs are characterized in terms of the linear coefficients of the corresponding factorial models. Some statistical properties of the g.l.s.e. under procedures of choosing a fractional replication at random, are then studied. First we prove that there is no g.l.s.e. which yields unbiased estimates of the entire vector of 2^m parameters. However, there are g.l.s.e.s which estimate unbiasedly subvectors of parameters. The trace of the mean-square-error matrix corresponding to a g.l.s.e. applied under certain randomization procedure is used as a loss function for the decision problem of choosing a g.l.s.e. and a randomization procedure. It is shown that when the parameters of the factorial system may assume arbitrary values, the randomization procedure which assigns equal probabilities to various fractional

replications (denoted by $R.P.*$) is admissible. Bayes g.l.s.e., relative to a-priori information available on the parameters, are then studied. This leads to a minimax theorem, which specifies a minimax and admissible g.l.s.e. under $R.P.*$.

The relationship between the generalized inverse of the matrix of normal equations and g.l.s.e. as given by A. Ben-Israel and J. Wersen (1962), and by C. R. Rao (1962) is studied. It is shown that these are particular cases in the general class of g.l.s.e. studied presently.

Finally, it should be remarked that although the present paper deals with factorial system of order 2^m all the important results hold in more general factorial systems of order p^m ($p > 2$).

2. The statistical model for fractional replication designs.

2.a. The statistical model for a full factorial experiment of order 2^m .

A full factorial experiment of order 2^m is a set of 2^m treatment combinations, consisting of m factors X_0, \dots, X_{m-1} each at two levels. Such a system can be characterized by 2^m parameters $\alpha_0, \dots, \alpha_{2^m-1}$, which are the coefficients of the multilinear regression function:

$$(2.1) \quad E\{Y(X_0, \dots, X_{m-1})\} = \sum_{(\lambda_0, \dots, \lambda_{m-1})} \alpha_{u(\lambda_0, \dots, \lambda_{m-1})} X_0^{\lambda_0} X_1^{\lambda_1} \dots X_{m-1}^{\lambda_{m-1}}$$

where $\lambda_j = 0, 1$ ($j=0, \dots, m-1$); $u(\lambda_0, \dots, \lambda_{m-1}) = \sum_{j=0}^{m-1} \lambda_j 2^j$;

and $Y(X_0, \dots, X_{m-1})$ is a random variable representing the "yield" of the experiment at treatment combination (X_0, \dots, X_{m-1}) . Denote by $X_{j,0}$ and $X_{j,1}$ ($j=0, \dots, m-1$), $X_{j,0} < X_{j,1}$, the two specified levels of factor X_j . By changing variables according to the transformation

$$(2.2) \quad Z_{j,k} = \frac{X_{j,k} - \frac{1}{2}(X_{j,0} + X_{j,1})}{\frac{1}{2}(X_{j,1} - X_{j,0})} \quad (k=0,1; j=0, \dots, m-1)$$

the regression function (2.1) is reduced to the form:

$$(2.3) \quad E\{Y(Z_0, \dots, Z_{m-1})\} = \sum_{u=0}^{2^m-1} \beta_u Z_0^{\lambda_0} Z_1^{\lambda_1} \dots Z_{m-1}^{\lambda_{m-1}}$$

where $Z_j = -1, +1$ ($j=0, \dots, m-1$); and $u \equiv u(\lambda_0, \dots, \lambda_{m-1})$.

Writing $z_j = (-1)^{1-i_j}$ with $i_j = 0, 1$ for all $j = 0, \dots, m-1$, the regression function (2.3) can be represented in the form:

$$(2.4) \quad E(Y(i_0, \dots, i_{m-1})) = \sum_{u=0}^{2^m-1} \beta_u (-1)^{j=0} \sum_{j=0}^{m-1} \lambda_j (1-i_j)$$

Furthermore, denote by $x_v \equiv (i_0, \dots, i_{m-1})$, $v = \sum_{j=0}^{m-1} i_j 2^j$, the 2^m treatment combinations of the factorial system under consideration; and let

$$(2.5) \quad c_{vu}^{(2^m)} = (-1)^{j=0} \sum_{j=0}^{m-1} \lambda_j (1-i_j), \quad \text{for all } v, u = 0, \dots, 2^m-1$$

then (2.4) is reduced to the form:

$$(2.5) \quad E(Y(x_v)) = \sum_{u=0}^{2^m-1} \beta_u c_{vu}^{(2^m)}, \quad \text{for all } v = 0, \dots, 2^m-1$$

Let $Y' = (Y(X_0), \dots, Y(X_{2^m-1}))$ be the vector of observations at all the 2^m treatment combinations; and let $\beta' = (\beta_0, \dots, \beta_{2^m-1})$ be the vector of parameters of (2.5). Thus, if

$$(C^{(2^m)}) = \|c_{vu}^{(2^m)}\|, \quad (v, u = 0, \dots, 2^m-1),$$

denotes the matrix of the coefficients of the β 's in (2.5), then the statistical model for a full factorial system can be written as:

$$(2.6) \quad Y = (C^{(2^m)}) \beta + \epsilon$$

where ϵ is a random vector, with $E \epsilon = 0$ and $E \epsilon \epsilon' = \sigma^2 I^{(2^m)}$ ($I^{(n)}$ denoting the identity matrix of order n).

2.b. The algebra of factorial experiments.

2.b.1. Properties of the matrices $(C^{(2^m)})$

The properties of the matrices $(C^{(2^m)})$ will be presented without proofs. For details see S. Ehrenfeld and S. Zacks (1961).

The matrices $(C^{(2^m)})$, $m = 1, 2, \dots$, of (2.6) can be obtained recursively by the following relationship:

$$(2.7) \quad (C^{(2^m)}) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \otimes (C^{(2^{m-1})}), \quad m = 1, 2, \dots$$

where $(C^{(0)}) \equiv 1$ (scalar), and where $A \otimes B$ is the Kronecker's direct multiplication of the matrix A by the matrix B from the left, defined as follows: If A is an $n \times m$ matrix $\|a_{ij}\|$, and B is a $k \times l$ matrix $\|b_{rs}\|$, then $A \otimes B$ is the $nk \times ml$ matrix:

$$A \otimes B = \begin{bmatrix} a_{11} B & \dots & a_{1m} B \\ \vdots & \ddots & \vdots \\ a_{n1} B & \dots & a_{nm} B \end{bmatrix}$$

From the associative property of the Kronecker's direct multiplication it follows that every matrix $(C^{(2^m)})$ can be factorized into $2^{m-s} \times 2^{m-s}$ ($1 \leq s \leq m$) submatrices of order $2^s \times 2^s$, according to the relationship:

$$(2.8) \quad (C^{(2^m)}) = (C^{(2^{m-s})}) \otimes (C^{(2^s)}), \quad (1 \leq s \leq m)$$

From this relationship it follows that the elements of $(C^{(2^m)})$ are related to those of $(C^{(2^{m-s})})$ and $(C^{(2^s)})$ according to:

$$(2.9) \quad C_{i+j2^s, t}^{(2^m)} = C_{i, r_t}^{(2^s)} \cdot C_{j, q_t}^{(2^{m-s})}$$

for all $i = 0, \dots, 2^s-1$; $j = 0, \dots, 2^{m-s}-1$; and where $t = r_t + q_t 2^s$ ($r_t = 0, \dots, 2^s-1$; $q_t = 0, \dots, 2^{m-s}-1$).

Another useful relationship among the elements of $(C^{(2^m)})$ is the following one: Let $u_1, u_2 = 0, \dots, 2^m-1$ be given by $u_1 \equiv (\lambda_{01}, \lambda_{11}, \dots, \lambda_{(m-1)1})$ ($i=1, 2$); $\lambda_{j,1} = 0, 1$ ($j=0, \dots, m-1$), and define $u_1 \oplus u_2 \equiv (\lambda'_0, \dots, \lambda'_{m-1})$ where $\lambda'_j \equiv \lambda_{j1} + \lambda_{j2} \pmod{2}$; then

$$(2.10) \quad C_{v, u_1 \oplus u_2}^{(2^m)} = C_{v, u_1}^{(2^m)} \cdot C_{v, u_2}^{(2^m)} \quad \text{for every } v=0, \dots, 2^m-1.$$

The properties of $(C^{(2)})$ are extended into $(C^{(2^m)})$ by the recursion relationship (2.7), and are summarized as follows:

- (i) $C_{v, 0}^{(2^m)} = 1$ for every $v = 0, \dots, 2^m-1$
- (ii) $C_{2^m-1, u}^{(2^m)} = 1$ for every $u = 0, \dots, 2^m-1$
- (iii) $\sum_{u=0}^{2^m-1} C_{vu}^{(2^m)} = 0$ for every $v = 0, \dots, 2^m-2$
- (iv) $\sum_{v=0}^{2^m-1} C_{vu}^{(2^m)} = 0$ for every $u = 1, \dots, 2^m-1$ for every $u = 1, \dots, 2^m-1$.

$$(v) \sum_{v=0}^{2^m-1} c_{vu_1}^{(2^m)} c_{vu_2}^{(2^m)} = \begin{cases} 2^m, & \text{if } u_1 = u_2 \\ 0, & \text{if } u_1 \neq u_2 \end{cases}$$

and

$$(vi) \sum_{u=0}^{2^m-1} c_{v_1 u}^{(2^m)} c_{v_2 u}^{(2^m)} = \begin{cases} 2^m, & \text{if } v_1 = v_2 \\ 0, & \text{if } v_1 \neq v_2 \end{cases}$$

(v) and (vi) can be expressed also in the form:

$$(c^{(2^m)})' (c^{(2^m)}) = (c^{(2^m)})(c^{(2^m)})' = 2^m I^{(2^m)}.$$

2.b.2. The group of parameters.

Every parameter β_u ($u = 0, \dots, 2^m-1$) of the statistical model (2.6) can be represented by an m -tuple $\beta_u \equiv (\lambda_0, \dots, \lambda_{m-1})$ where $\lambda_j = 0, 1$ ($j = 0, \dots, m-1$).

The set of all 2^m parameters constitutes a group, B with respect to the operator \otimes , defined as follows:

$$\text{Let } u_1 = \sum_{j=0}^{m-1} \lambda_j 2^j \text{ and } u_2 = \sum_{j=0}^{m-1} \lambda'_j 2^j \text{ then } \beta_k \equiv \beta_{u_1} \otimes \beta_{u_2}$$

$$\text{if, and only if } k = \sum_{j=0}^{m-1} \lambda''_j 2^j \text{ where } \lambda''_j \equiv \lambda_j + \lambda'_j \pmod{2} \text{ for}$$

all $j = 0, \dots, m-1$.

The unit element of the group B is $\beta_0 \equiv (0, 0, \dots, 0)$ and the inverse of $\beta_u \equiv (\lambda_0, \dots, \lambda_{m-1})$ is $\beta_u^{-1} \equiv (2-\lambda_0, 2-\lambda_1, \dots, 2-\lambda_{m-1}) \pmod{2}$. A set of n parameters $\beta_{u_1}, \beta_{u_2}, \dots, \beta_{u_n}$ is called dependent if there

exist n constants a_k ($k=1, \dots, n$), not all of which are zero ($a_k = 0, 1$), such that:

$$(2.11) \quad [\beta_{u_1}]^{a_1} \otimes [\beta_{u_2}]^{a_2} \otimes \dots \otimes [\beta_{u_n}]^{a_n} \equiv \beta_0$$

where

$$[\beta_u]^a = \begin{cases} \beta_u & , \text{ if } a = 1 \\ \beta_0 & , \text{ if } a = 0 \end{cases}$$

If relationship (2.11) is valid only when all $a_k = 0$ ($k=1, \dots, n$) then $\beta_{u_1}, \dots, \beta_{u_n}$ are called independent. It is easy to check that every set of n independent parameters ($1 \leq n \leq m$) generates a subgroup of order 2^n in B .

2.c. The construction of a $1/2^{m-s}$ ($1 \leq s, m$) fractional replication.

Let $\{\beta_{d_0}, \beta_{d_1}, \dots, \beta_{d_{m-s-1}}\}$ be any set of $m-s$ independent parameters in B . The 2^m treatment combinations can be classified into 2^{m-s} disjoint subsets X_v ($v=0, \dots, 2^{m-s}-1$) of equal size, relative to the specified $m-s$ independent parameters, in the following manner: Let $\beta_{d_k} \equiv (\lambda_{0,d_k}, \lambda_{1,d_k}, \dots, \lambda_{m-1,d_k})$ $k = 0, \dots, m-s-1$ be a defining parameter, and let $X \equiv (i_0, \dots, i_{m-1})$ be a treatment combination, then $x \in X_v$ if, and only if,

$$(2.12) \quad \sum_{j=0}^{m-1} i_j \lambda_{j,d_k} \equiv a_k \pmod{2}, \text{ and where}$$

$$v = \sum_{k=0}^{m-s-1} a_k 2^k.$$

In order to perform the classification of treatment combinations into the blocks X_v ($v = 0, \dots, 2^{m-s}-1$) we do not have to solve the linear equations (2.12), but it suffices to compare the rows of the matrix of coefficients $(c^{(2^m)})$ under the columns corresponding to the special independent parameters $(\beta_{d_0}, \dots, \beta_{d_{m-s-1}})$. These two procedures are equivalent (see S. Ehrenfeld and S. Zacks (1961)). The $m-s$ independent parameters, relative to which the classification takes place, are called defining parameters. The answer to the question, which of the parameters should be specified for the role of defining ones depends on the objectives of the experiment. The choice of a set of defining parameters will generally effect the bias of estimators and their variances, and might have other effects on the properties of statistics and procedures (see O. Kempthorne (1952); S. Ehrenfeld and S. Zacks (1961)).

The term fractional replication, in its broadest sense, relates to any subset of treatment combination from a full factorial system. K. Takeuchi (1961) considers designs of randomly combined fractional replications. We shall consider in the present paper only fractional replications which consist of one block of treatment combinations, X_v , chosen from the set of 2^{m-s} blocks constructed according to the procedure outlined above. These fractional replications are called orthogonal. A randomized fractional replication procedure is one in which a block X_v is chosen with a probability vector $\xi' = (\xi_0, \dots, \xi_{2^{m-s}-1})$

2.d. The statistical model for a $1/2^{m-s}$ fractional replication.

Let $(\beta_{d_0}, \dots, \beta_{d_{m-s-1}})$ be a set of defining parameters;
 $(X_v; v = 0, \dots, 2^{m-s}-1)$ the corresponding blocks of treatment combinations,

and $Y(X_v)$ the vector of observations associated with the 2^s treatment combinations in X_v . The order of the components of $Y(X_v)$ is determined by the standard order of the corresponding x 's in X_v , e.g. if $X_v = \{x_0, x_3, x_5, x_6\}$ then $Y(X_v)' = (Y(x_0), Y(x_3), Y(x_5), Y(x_6))$.

Let $\beta^{(0)'} = (\beta_0, \beta_{t_1}, \dots, \beta_{t_{2^s-1}})$ be any specified vector of 2^s parameters independent of the defining parameters (except for the "mean" β_0); with $t_k < t_{k+1}$ for all $k = 1, \dots, 2^s-1$. Let $\{\beta_u^*; u=0, \dots, 2^{m-s}-1\}$ be the subgroup of 2^{m-s} parameters, generated by the $m-s$ defining parameters. Define by $\beta^{(u)}$ ($u = 1, \dots, 2^{m-s}-1$) the vector of 2^s parameters obtained by multiplying each of the components of $\beta^{(0)}$ by β_u^* , i.e., $\beta^{(u)} = (\beta_0 \otimes \beta_u^*, \beta_{t_1} \otimes \beta_u^*, \dots, \beta_{t_{2^s-1}} \otimes \beta_u^*)'$. Then, the statistical model for $Y(X_v)$ can be written in the form:

$$(2.13) \quad Y(X_v) = \sum_{u=0}^{2^{m-s}-1} (P_{vu}^{(2^s)}) \beta^{(u)} + \epsilon = (P_v) \beta^* + \epsilon$$

where; as proven by S. Ehrenfeld and S. Zacks (1961)

$$(2.14) \quad (P_v) = (1, b_{v1}^{(2^{m-s})}, \dots, b_{v(2^{m-s}-1)}^{(2^{m-s})}) \otimes (P_{v0}^{(2^s)})$$

is a $2^s \times (2^m - 2^s)$ matrix; $(P_{v0}^{(2^s)})$ is a $2^s \times 2^s$ matrix obtained from $(C^{(2^m)})$, by picking the elements of $(C^{(2^m)})$ corresponding to treatment combinations in X_v and the parameters in $\beta^{(0)}$, and arranging them in the standard order. The scalars $b_{vu}^{(2^{m-s})}$, by which we multiply $(P_{v0}^{(2^s)})$ to obtain $(P_{vu}^{(2^s)})$, are given by the formula:

$$(2.15) \quad b_{vu}^{(2^{m-s})} = (-1)^{\sum_{j=0}^{m-s-1} (i_j' (i_j - L(d_j)))}$$

where $v = \sum_{j=0}^{m-s-1} i_j 2^j$; $u = \sum_{j=0}^{m-s-1} i'_j 2^j$ and $L(d_j) \equiv \sum_{k=0}^{m-1} \lambda_{k,d_j} \pmod{2}$,

for every defining parameter β_{d_j} ; and where $\beta^* = (\beta^{(0)} : \beta^{(1)} : \dots : \beta^{(2^{m-s}-1)})'$,
and ϵ is a random vector of order 2^s , with $E\epsilon = 0$ and $E\epsilon\epsilon' = \sigma^2 I^{(2^s)}$.

It can be readily proved that, the rows of $(P_{vu}^{(2^s)})$ ($v, u=0, \dots, 2^{m-s}-1$),
as well as its columns, are orthogonal, i.e.,

$$(2.16) \quad (P_{vu}^{(2^s)})' (P_{vu}^{(2^s)}) = (P_{vu}^{(2^s)}) (P_{vu}^{(2^s)})' = 2^s I^{(2^s)}$$

for all $v, u=0, \dots, 2^{m-s}-1$; and that a similar property holds for the
matrix $(B_{(d_s, \dots, d_{m-s-1})}^{(2^{m-s})})$, whose elements are the coefficients $b_{vu}^{(2^{m-s})}$,
defined by (2.15), i.e.,

$$(2.17) \quad (B_{(d_0, \dots, d_{m-s-1})}^{(2^{m-s})})' (B_{(d_0, \dots, d_{m-s-1})}^{(2^{m-s})}) = 2^{m-s} I^{(2^{m-s})}$$

for every choice of $m-s$ defining parameters. For the sake of simpli-
fying notation, let $S = 2^s$ and $M = 2^{m-s}$, $N = S \cdot M = 2^m$. Furthermore,
assume, without loss of generality, that the defining parameters are the
"main effects" $(\beta_S, \beta_{2S}, \dots, \beta_{N/2})$ and that $\beta^{(0)} = (\beta_0, \beta_1, \dots, \beta_{S-1})'$
then, the blocks of treatment combinations are:

$$(2.18) \quad \{X_{i+vS}; i = 0, \dots, S-1\} \text{ for all } v = 0, \dots, M-1$$

and the statistical model for $Y(X_v)$ is given by:

$$(2.19) \quad Y(X_v) = [(1, c_{v1}^{(M)}, \dots, c_{v(M-1)}^{(M)}) \otimes (c^{(S)})] \beta + \epsilon \\ = (c^{(S)}) \beta^{(0)} + \sum_{u=1}^{M-1} c_{vu}^{(M)} \beta^{(u)} + \epsilon$$

where $\beta^{(u)} = (\beta_{us}, \beta_{1+us}, \dots, \beta_{(u+1)s-1})'$.

3. Generalized least squares estimators for fractional replications.

3.a. The set of all least squares estimators.

Given a block of treatment combinations X_v ($v=0, \dots, M-1$) and the associated vector of observations $Y(X_v)$, the "normal equations" corresponding to the linear model (2.19) are given by:

$$(3.1) \quad (C_v)' (C_v) \beta = (C_v)' Y(X_v), \quad v=0, \dots, M-1,$$

where (C_v) is the $S \times N$ matrix of the coefficients of (2.19), i.e.,

$$(3.2) \quad (C_v) = (1, c_{v1}^{(M)}, \dots, c_{v(M-1)}^{(M)}) \otimes (C^{(S)}).$$

A generalized least squares estimator (g.l.s.e.) of β is any linear operator (L_v) , on $E^{(S)}$ (Eucliden S -space), so that (L_v) is an $N \times S$ matrix satisfying the equation:

$$(3.3) \quad (C_v)' (C_v) (L_v) = (C_v)', \quad v=0, \dots, M-1.$$

Let $(L_v)' = ((L_{v0})' : (L_{v1})' : \dots : (L_{v(M-1)})')$ where (L_{vu}) ($u=0, \dots, M-1$) are square matrices of order $S \times S$. Substituting from (3.2) for (C_v) in 3.3 and decomposing (L_v) as indicated we arrive at the matrix equation:

$$(3.4) \quad S[(Q^{(M)}) \otimes (I^{(S)})] \begin{bmatrix} (L_{v0}) \\ \vdots \\ (L_{v(M-1)}) \end{bmatrix} = \begin{bmatrix} 1 \\ c_{v1}^{(M)} \\ \vdots \\ c_{v(M-1)}^{(M)} \end{bmatrix} \otimes (C^{(S)}),$$

where $(Q^{(M)}) = (1, c_{v1}^{(M)}, \dots, c_{v(M-1)}^{(M)})' (1, c_{v1}^{(M)}, \dots, c_{v(M-1)}^{(M)})$ is a

square symmetric matrix of order $M \times M$, whose (i,j) -th element is $q_{ij}^{(M)} = c_{vi}^{(M)} c_{vj}^{(M)}$ ($i,j=0,\dots,M-1$). Since $c_{vu}^{(M)} = \pm 1$ and $(C^{(S)})$ is non-singular, the linear equations in the matrices (L_{vu}) can be expressed in a form equivalent to (3.4) as,

$$(3.5) \quad \sum_{u=0}^{M-1} c_{vu}^{(M)} (L_{vu}) (C^{(S)}) = I^{(S)}.$$

Since the unique solution to the equation $(H)(C^{(S)}) = I^{(S)}$ is $(H) = \frac{1}{S} (C^{(S)})'$, it follows that the M matrices $(L_v^{(j)})$, $j=0,\dots,M-1$, whose submatrices are given by:

$$(3.6) \quad (L_{vu}^{(j)}) = \begin{cases} \frac{1}{S} c_{vu}^{(M)} (C^{(S)})', & \text{if } u = j-1 \\ (0) & \text{, otherwise} \end{cases}$$

constitute a basis of M independent solutions of (3.3). Thus, every g.l.s.e. (L_v) can be represented as a linear combination of the M linearly independent operators $(L_v^{(j)})$ ($j=0,\dots,M-1$), so that the coefficients of $(L_v^{(j)})$ add up to 1. Formally, the set of all g.l.s.e., given X_v is:

$$(3.7) \quad \mathcal{L}(C_v) = \{(L_v) : (L_v) = \sum_{j=0}^{M-1} \lambda_j (L_v^{(j)}) ; \sum_{j=0}^{M-1} \lambda_j = 1\}$$

Every g.l.s.e. can thus be represented by M coordinates

$(\lambda_0, \lambda_1, \dots, \lambda_{M-1})$ such that $\sum_{j=0}^{M-1} \lambda_j = 1$. Furthermore, if $\hat{\beta}_v$ de-

notes the vector of g.l.s.e. of β then we have, according

to (3.6) and (3.7)

$$(3.8) \quad \hat{\beta}_v = \frac{1}{S} \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_{M-1} \end{bmatrix} C_{vu}^{(M)} \otimes (C^{(S)})' Y(X_v) \text{ for every } v=0, \dots, M-1.$$

3.b. Some statistical properties of g.l.s.e.

In the present section we prove that there are no unbiased g.l.s.e. of β , and derive an expression for the trace of the mean-square-error matrix of a g.l.s.e. $\hat{\beta}$.

Consider a fractional replication design in which a block X_v ($v=0, \dots, M-1$) is chosen with probability ξ_v ($\xi_v \geq 0$ for all $v=0, \dots, M-1$; $\sum_{v=0}^{M-1} \xi_v = 1$). A randomized fractional replication procedure is thus represented by an M-dimensional probability vector ξ . This class of randomization procedures contains, in particular, the fixed fractional replication design, in which one of the X_v blocks is chosen with probability one.

Theorem 1.

Let $\tilde{\beta}^{(u)} = \frac{1}{S} C_{vu}^{(M)} (C^{(S)})' Y(X_v)$ then $E_{\xi^*} \{\tilde{\beta}^{(u)}\} = \beta^{(u)}$ for all $u=0, \dots, M-1$ if, and only if, $\xi^* = (\frac{1}{M}, \frac{1}{M}, \dots, \frac{1}{M})'$.

Proof.

The expected value of $\tilde{\beta}^{(u)}$ under randomization procedure ξ , is given by:

$$\begin{aligned}
(3.9) \quad E_{\xi} \{ \tilde{\beta}^{(u)} \} &= \frac{1}{S} \sum_{v=0}^{M-1} \xi_v C_{vu}^{(M)} (C^{(S)})', Y(X_v) \\
&= \frac{1}{S} \sum_{v=0}^{M-1} \xi_v C_{vu}^{(M)} (C^{(S)})', \left[\sum_{w=0}^{M-1} C_{vw}^{(M)} (C^{(S)}) \beta^{(w)} + E\{\epsilon\} \right] \\
&= \sum_{w=0}^{M-1} \left(\sum_{v=0}^{M-1} \xi_v C_{vu}^{(M)} C_{vw}^{(M)} \right) \beta^{(w)} = \beta^{(u)} + \sum_{(w \neq u)=0}^{M-1} \sum_{v=0}^{M-1} \xi_v \cdot \\
&\quad \cdot C_{vu}^{(M)} C_{vw}^{(M)} \beta^{(w)}.
\end{aligned}$$

Clearly, if $\xi_v = \frac{1}{M}$ for all $v=0, \dots, M-1$, then

$$\sum_{v=0}^{M-1} \xi_v C_{vu}^{(M)} C_{vw}^{(M)} = \frac{1}{M} \sum_{v=0}^{M-1} C_{vu}^{(M)} C_{vw}^{(M)} = 0$$

for all $u \neq w$ by the orthogonality of the column vectors of $(C^{(M)})$.

Thus, $E_{\xi} \{ \tilde{\beta}^{(u)} \} = \beta^{(u)}$ for all $u=0, \dots, M-1$. On the other hand,

if $E_{\xi} \{ \tilde{\beta}^{(u)} \} = \beta^{(u)}$ for all $u=0, \dots, M-1$ then, in particular

$$(3.10) \quad E_{\xi} \{ \tilde{\beta}^{(0)} \} = \beta^{(0)} + \sum_{w=1}^{M-1} \sum_{v=0}^{M-1} \xi_v C_{vu}^{(M)} \beta^{(w)}$$

But the condition $\sum_{v=0}^{M-1} \xi_v C_{vu}^{(M)} = 0$ for all $w=1, \dots, M-1$ is equivalent to the condition:

$$(3.11) \quad (C^{(M)})' \xi = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Multiplying both sides of (3.11) by $(C^{(M)})$ we get:

$$(3.12) \quad M_{\xi} = (C^{(M)})(C^{(M)})', \quad \xi = (C^{(M)}) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 1^{(M)}$$

where $1^{(M)}$ is an M-dimensional vector with unity in all its components.

It follows that a necessary condition for the unbiasedness of $\tilde{\beta}^{(u)}$ is that $\xi = \frac{1}{M} 1^{(M)}$, i.e., each block is chosen with the same probability.

(Q.E.D.)

Returning to the g.l.s.e. we have:

$$(3.13) \quad E_{\xi^*} \{\hat{\beta}\} = E_{\xi^*} \{(\lambda_0 \tilde{\beta}^{(0)'} : \lambda_1 \tilde{\beta}^{(1)'} : \dots : \lambda_{M-1} \tilde{\beta}^{(M-1)'})'\}$$

$$= (\lambda_0 \beta^{(0)'} : \lambda_1 \beta^{(1)'} : \dots : \lambda_{M-1} \beta^{(M-1)'})',$$

where $\xi^* = \frac{1}{M} 1^{(M)} = \frac{1}{M} (1, 1, \dots, 1)'$.

Since $\sum_{u=0}^{M-1} \lambda_u = 1$ we conclude that there is no unbiased g.l.s.e. of β .

The g.l.s.e. in which $\lambda_0 = 1$ and $\lambda_u = 0$ for all $u > 0$ yields unbiased estimates of the components of $\beta^{(0)}$ only. Similarly when $\lambda_j = 1$ ($j=0, \dots, M-1$) and $\lambda_{j'} = 0$ for all $j' \neq j$, the corresponding g.l.s.e. yields unbiased estimates of the components of $\beta^{(j)}$ only.

The mean-square-error dispersion matrix of a g.l.s.e. $\hat{\beta}$, under randomization procedure ξ , is defined by $E_{\xi} \{(\hat{\beta} - \beta)(\hat{\beta} - \beta)'\}$. Let $M(\xi, \lambda; \beta)$ denote the trace of the mean-square-error dispersion matrix of a g.l.s.e. represented by a vector λ , such that $\lambda' 1^{(M)} = 1$, under randomization procedure ξ ; i.e., $M(\xi, \lambda; \beta) = E_{\xi} \{(\hat{\beta} - \beta)' (\hat{\beta} - \beta)\}$.

Theorem 2.

The trace of the mean-square-error dispersion matrix under randomization procedure ξ is given by the expression:

$$(3.14) \quad M(\xi, \lambda; \beta) = (\sigma^2 + |\beta|^2) \sum_{u=0}^{M-1} \lambda_u^2 - \sum_{u=0}^{M-1} (2\lambda_u - 1) |\beta^{(u)}|^2 + \\ + \sum_{u_1=0}^{M-1} (2\lambda_{u_1} + 1) \sum_{(u_2 \neq u_1)=0}^{M-1} \left[\sum_{v=0}^{M-1} \xi_v c_{vu_1}^{(M)} c_{vu_2}^{(M)} \right] \beta^{(u_1)'} \beta^{(u_2)}$$

where $|\beta|^2 = \beta' \beta$, $|\beta^{(u)}|^2 = \beta^{(u)'} \beta^{(u)}$ ($u=0, \dots, M-1$).

Proof.

$$(3.15) \quad M(\xi, \lambda, \beta) = E_{\xi} \{ (\hat{\beta} - \beta)' (\hat{\beta} - \beta) \} = E_{\xi} \{ \hat{\beta}' \hat{\beta} \} - 2 \beta' E_{\xi} \{ \hat{\beta} \} + \beta' \beta$$

According to (3.8)

$$(3.16) \quad E_{\xi} \{ \hat{\beta}' \hat{\beta} \} = \frac{1}{S} \sum_{u=0}^{M-1} \lambda_u^2 E_{\xi} \{ Y(X_u)' Y(X_u) \}$$

Substituting (2.19) for $Y(X_u)$ in (3.16) we get:

$$(3.17) \quad \frac{1}{S} E_{\xi} \{ Y(X_u)' Y(X_u) \} = \frac{1}{S} E_{\xi} \left\{ \left[\sum_{u_1=0}^{M-1} c_{vu_1}^{(M)} (C^{(S)})_{\beta}^{(u_1)} + \epsilon \right]' \cdot \right. \\ \cdot \left. \left[\sum_{u_2=0}^{M-1} c_{vu_2}^{(M)} (C^{(S)})_{\beta}^{(u_2)} + \epsilon \right] \right\} = \\ = \sigma^2 + \frac{1}{S} E_{\xi} \left\{ \sum_{u_1=0}^{M-1} \sum_{u_2=0}^{M-1} c_{vu_1}^{(M)} c_{vu_2}^{(M)} \beta^{(u_1)'} (C^{(S)})' (C^{(S)})_{\beta}^{(u_2)} \right\} \\ = \sigma^2 + \sum_{u=0}^{M-1} |\beta^{(u)}|^2 + \sum_{u_1 \neq u_2} \left(\sum_{v=0}^{M-1} \xi_v c_{vu_1}^{(M)} c_{vu_2}^{(M)} \right) \beta^{(u_1)'} \beta^{(u_2)}$$

Furthermore, $E_{\xi}(\hat{\beta}') = (\lambda_0 E_{\xi}(\tilde{\beta}^{(0)'}) \dots \lambda_{M-1} E_{\xi}(\tilde{\beta}^{(M-1)'}))$

Substituting (3.9) for $E_{\xi}(\tilde{\beta}^{(u)})$, we arrive at

$$(3.18) \quad \beta' E_{\xi}(\hat{\beta}) = \sum_{u=0}^{M-1} \lambda_u |\beta^{(u)}|^2 + \sum_{u_1=0}^{M-1} \lambda_{u_1} \left[\sum_{u_2 \neq u_1}^{M-1} \left(\sum_{v=0}^{M-1} \xi_v c_{vu_1}^{(M)} c_{vu_2}^{(M)} \right) \cdot \beta^{(u_1)} \beta^{(u_2)} \right]$$

Thus, from (3.15)-(3.18) the result holds.

Q.E.D.

Corollary: When each block X_v ($v=0, \dots, M-1$) is chosen with equal probabilities ($\xi = \xi^*$) we have

$$(3.19) \quad M(\xi^*, \lambda; \beta) = (\sigma^2 + |\beta|^2) \sum_{u=0}^{M-1} \lambda_u^2 - \sum_{u=0}^{M-1} (2\lambda_u - 1) |\beta^{(u)}|^2$$

3.c. Optimum strategies

A strategy of the Statistician is a pair of two M-dimensional vectors (ξ, λ) such that ξ is a probability vector, and $\lambda' 1^{(M)} = 1$. Every strategy (ξ, λ) represents a randomization procedure and a g.l.s.e. The decision problem is to choose (ξ, λ) optimally, with respect to the loss function $M(\xi, \lambda; \beta)$.

Comparing (3.14) to (3.19) it is easily verified that for every (ξ, λ) there exists β^0 in E^n such that $M(\xi, \lambda; \beta^0) > M(\xi^*, \lambda; \beta^0)$.

Thus, whenever β is arbitrary, ξ^* represents an admissible randomization procedure. For this reason we shall restrict the discussion from now on to strategies with randomization procedure ξ^* , and turn now to the problem of deciding upon an optimum g.l.s.e. under ξ^* .

We notice in (3.19) that $M(\xi^*, \lambda; \beta)$ depends on β only through the M values $|\beta^{(u)}|^2$. An a-priori information concerning these values might thus be utilized for the choice of λ . Thus, let $\Pi^{(u)}$ be an a-priori distribution of $|\beta^{(u)}|^2$, defined over the half-line $[0, \infty)$.

Theorem 3.

The Bayes g.l.s.e. of β , with respect to the a-priori distributions $\{\Pi^{(0)}, \dots, \Pi^{(M-1)}\}$, under randomization procedure ξ^* is determined by the vector $\lambda_\pi = (\lambda_\pi^{(0)}, \dots, \lambda_\pi^{(M-1)})$, where

$$(3.20) \quad \lambda_\pi^{(u)} = \frac{E_{\pi^{(u)}}\{|\beta^{(u)}|^2\}}{\sum_{u=0}^{M-1} E_{\pi^{(u)}}\{|\beta^{(u)}|^2\}}, \quad \text{for all } u=0, \dots, M-1.$$

Proof:

The risk function under (ξ^*, λ) and Π is

$$(3.21) \quad R(\xi^*, \lambda; \Pi) = (\sigma^2 + \sum_{u=0}^{M-1} E_{\pi^{(u)}}\{|\beta^{(u)}|^2\}) \sum_{u=0}^{M-1} \lambda_u^2 - \sum_{u=0}^{M-1} (2\lambda_u - 1) E_{\pi^{(u)}}\{|\beta^{(u)}|^2\}.$$

It is easily verified that $\lambda_\pi^{(u)}$ ($u=0, \dots, M-1$), given by (3.20), minimize (3.21) under the constraint $\sum_{u=0}^{M-1} \lambda_u = 1$.

Q.E.D.

Let $R_\pi^{(u)} = E_{\pi^{(u)}}\{|\beta^{(u)}|^2\}$ ($u=0, \dots, M-1$) and $R_\pi = \sum_{u=0}^{M-1} R_\pi^{(u)}$ then the Bayes risk with respect to an a-priori distribution Π is

$$(3.22) \quad R(\xi^*, \lambda_\pi; \Pi) = (\sigma^2 + R_\pi) \sum_{u=0}^{M-1} (\lambda_\pi^{(u)})^2 - \\ - \sum_{u=0}^{M-1} (2\lambda_\pi^{(u)} - 1) R_\pi^{(u)} = R_\pi - \frac{1}{R_\pi} \left(1 - \frac{\sigma^2}{R_\pi}\right) \sum_{u=0}^{M-1} (R_\pi^{(u)})^2$$

In particular, when all $|\beta^{(u)}|^2$ ($u=0, \dots, M-1$) have the same a-priori distribution, with $R_\pi^{(u)} = R_\pi^*$ for all $u=0, \dots, M-1$, then the Bayes g.l.s.e. is represented by $\lambda^* = (\frac{1}{M}, \frac{1}{M}, \dots, \frac{1}{M})$ with a Bayes risk

$$(3.23) \quad R(\xi^*, \lambda^*, \Pi) = \frac{\sigma^2}{M} + (M-1)R_\pi^*$$

Theorem 4.

$\lambda^* = \frac{1}{M} \mathbf{1}^{(M)}$ represents the minimax and admissible g.l.s.e. under randomization procedure ξ^* relative to the class of all a-priori distributions Π , such that $R_\pi = \sum_{u=0}^{M-1} R_\pi^{(u)} = \text{const.}$ and $(\sigma^2 < R_\pi < \infty)$. The minimax risk is given by (3.23).

Proof:

The minimax risk is the maximal Bayes risk, with respect to all the a-priori distributions Π in the class considered. The Bayes risk for any of these π 's is given by (3.22) where $R_\pi = \sum_{u=0}^{M-1} R_\pi^{(u)}$ is a given constant. Set the Lagrangian

$$(3.24) \quad L(R_\pi^{(0)}, \dots, R_\pi^{(M-1)}; \rho) = R_\pi - \frac{1}{R_\pi} \left(1 - \frac{\sigma^2}{R_\pi}\right) \sum_{u=0}^{M-1} (R_\pi^{(u)})^2 + \\ + \rho \left(R_\pi - \sum_{u=0}^{M-1} R_\pi^{(u)}\right)$$

By differentiating partially with respect to $R_{\pi}^{(u)}$ ($u=0, \dots, M-1$) and ρ and equating the derivatives to zero we arrive at the system of linear equations:

$$(3.25) \quad \left\{ \begin{array}{l} -\frac{2}{R_{\pi}} \left(1 - \frac{\sigma^2}{R_{\pi}}\right) R_{\pi}^{(u)} = \rho \quad \text{for all } u=0, \dots, M-1 \\ \sum_{u=0}^{M-1} R_{\pi}^{(u)} = R_{\pi} \end{array} \right.$$

The solution of this system of linear equations is given by $R_{\pi}^{(u)} = \frac{R_{\pi}}{M}$ for every $u=0, \dots, M-1$. Furthermore, since $R_{\pi} \geq \sigma^2$, all the second order partial derivatives with respect to $R_{\pi}^{(u)}$ are negative. Thus all a-priori distributions π such that $R_{\pi}^{(u)} = \frac{R_{\pi}}{M}$ for every $u=0, \dots, M-1$ are minimax strategies for Nature. As mentioned before, $\lambda^* = \frac{1}{M} 1^{(M)}$ is then the unique minimax strategy for the Statistician. The Bayes risk corresponding to λ^* is given by (3.23). The admissibility of λ^* , relative to the class of a-priori distributions considered, follows from the fact that it is the unique minimax.

Q.E.D.

4. The generalized inverse and the g.l.s.e.

4.a. The g.l.s.e. of minimum norm

A. Ben-Israel and S. J. Wersan (1962) proved that the g.l.s.e. (L_v) with minimum norm, i.e., $\min_{(L_v)} \text{tr} (L_v)'(I_v)$, is a particular generalized inverse $(C_v)^\dagger$ of the matrix of coefficients in (2.19), namely $(C_v) = (1, c_{v1}^{(M)}, \dots, c_{v(M-1)}^{(M)}) \times (C^{(S)})$. The generalized inverse $(C_v)^\dagger$ always exists, it is unique, and given in general by the formula:

$$(4.1) \quad (C_v)^\dagger = [I^{(N)} - (D_v)(D_v' D_v)^{-1}(D_v)'](E_v)(C_v)'$$

for all $v=0, \dots, M-1$; where (E_v) is a product of elementary transformations, which transforms $(C_v)'(C_v)$ into:

$$(4.2) \quad (E_v)(X_v)'(X_v) = \begin{bmatrix} I^{(S)} & \vdots & (\Delta_v) \\ \dots & \vdots & \dots \\ (0) & \vdots & (0) \end{bmatrix} \quad (v=0, \dots, M-1)$$

and where

$$(4.3) \quad (D_v) = \begin{bmatrix} (\Delta_v) \\ . & . & . & . \\ -I^{(N-S)} \end{bmatrix} \quad (v=0, \dots, M-1) .$$

The generalized inverse matrix, $(C_v)^\dagger$ has the properties:

$$(4.4) \quad (C_v)(C_v)^\dagger(C_v) = (C_v) \quad \text{for all } v=0, \dots, M-1 .$$

and

$$(C_v)^\dagger(C_v)(C_v)' = (C_v)'$$

A straightforward computation of $(C_v)^\dagger$ according to formula (4.1) yields the result

$$(4.5) \quad (C_v)^\dagger = \frac{1}{M} \begin{bmatrix} \tilde{\beta}^{(0)} \\ \tilde{\beta}^{(1)} \\ \vdots \\ \tilde{\beta}^{(M-1)} \end{bmatrix}$$

That is, $(C_v)^\dagger$ is a g.l.s.e. represented by $\lambda^* = \frac{1}{M} \mathbf{1}^{(M)}$, and has the optimal properties mentioned in the previous section. This result can be obtained in the present framework more easily. According to (3.8) a g.l.s.e. is given by

$$(4.6) \quad (L_v) = \frac{1}{S} \begin{bmatrix} \lambda_o & C_{vo}^{(M)} \\ \vdots & \vdots \\ \lambda_{M-1} & C_{v(M-1)}^{(M)} \end{bmatrix} \otimes (C^{(S)})', \quad (v=0, \dots, M-1)$$

Accordingly, the norm of (L_v) is

$$(4.7) \quad \begin{aligned} \text{tr. } (L_v)'(L_v) &= \text{tr. } \left\{ \frac{1}{S^2} \sum_{u=0}^{M-1} \lambda_u^2 (C^{(S)})(C^{(S)})' \right\} \\ &= \frac{1}{S} \text{tr. } \left\{ \sum_{u=0}^{M-1} \lambda_u^2 I^{(S)} \right\} = \sum_{u=0}^{M-1} \lambda_u^2. \end{aligned}$$

Since the vector $\lambda^* = \frac{1}{M} \mathbf{1}^{(M)}$ minimizes $\sum_{u=0}^{M-1} \lambda_u^2$, under the constraint

$\sum_{u=0}^{M-1} \lambda_u = 1$, it follows that the g.l.s.e. represented by λ^* minimizes the norm of (L_v) , $(v=0, \dots, M-1)$.

4.b. The g.l.s.e. suggested by C. R. Rao.

C. R. Rao (1962) defines the g.l.s.e. of β by the operator $(L_v)^- = [(C_v)'(C_v)]^- (C_v)'$ where $[(C_v)'(C_v)]^-$ is a generalized inverse of $(C_v)'(C_v)$. In case $(C_v)'(C_v)$ is invertible, $(L_v)^-$ is the

unique g.l.s.e. of β . We shall prove now that under the present model of fractional replication designs, Rao's g.l.s.e., $(L_V)^-$, is represented by the vector $\lambda^- = (1, 0, 0, \dots, 0)$. For this purpose, define the matrix of elementary transformations

$$(4.8) \quad (E_V) = \frac{1}{\sqrt{S}} \begin{bmatrix} 1 & & & & \\ -C_{V1}^{(M)} & & & & 0 \\ \vdots & & 0 & \ddots & \\ -C_{V(M-1)}^{(M)} & & & & 1 \end{bmatrix} \otimes I^{(S)}$$

then we have, for every $v=0, \dots, M-1$,

$$(4.9) \quad (E_V)[(C_V)'(C_V)](E_V)' = \begin{bmatrix} I^{(S)} & \vdots & (0) \\ \vdots & \ddots & \vdots \\ (0) & \vdots & (0) \end{bmatrix}$$

Hence,

$$(4.10) \quad [(C_V)'(C_V)]^- = (E_V)'(E_V) \quad .$$

To show this, consider the relationship

$$(4.11) \quad [(C_V)'(C_V)][(C_V)'(C_V)]^-[(C_V)'(C_V)] = [(C_V)'(C_V)]$$

Multiply both sides of (4.11) from the left by (E_V) and from the right by $(E_V)'$. Then,

$$(4.12) \quad (E_V)[(C_V)'(C_V)](E_V)'(E_V)[(C_V)'(C_V)](E_V)' = (E_V)[(C_V)'(C_V)](E_V)'$$

Or according to (4.9)

$$(4.13) \quad \left[\begin{array}{c|c} I^{(S)} & (0) \\ \hline (0) & (0) \end{array} \right] \left[\begin{array}{c|c} I^{(S)} & (0) \\ \hline (0) & (0) \end{array} \right] = \left[\begin{array}{c|c} I^{(S)} & (0) \\ \hline (0) & (0) \end{array} \right] .$$

Accordingly,

$$(4.14) \quad (L_V)^- = (E_V)' (E_V) (C_V)'$$

$$= \frac{1}{S} \left[\begin{array}{cccc} M & -C_{v1}^{(M)} & \dots & -C_{v(M-1)}^{(M)} \\ -C_{v1}^{(M)} & & & \\ \vdots & 1 & \ddots & \\ -C_{v(M-1)}^{(M)} & & & 1 \end{array} \right] \left[\begin{array}{c} 1 \\ C_{v1}^{(M)} \\ \vdots \\ C_{v(M-1)}^{(M)} \end{array} \right] \otimes (C^{(S)}),$$

$$= \frac{1}{S} \left[\begin{array}{c} 1 \\ 0 \\ \vdots \\ 0 \end{array} \right] \otimes (C^{(S)}), = \left[\begin{array}{c} \tilde{\beta}^{(0)} \\ 0 \\ \vdots \\ 0 \end{array} \right]$$

That is, Rao's g.l.s.e. $(L_V)^-$ is represented by $\lambda^- = (1, 0, \dots, 0)'$.

From theorem 1 it follows that $(L_V)^-$ is an unbiased estimator of $(\beta^{(0)}, 0)$. The trace of the dispersion matrix of $(L_V)^- Y(X_V)$, under randomization procedure ξ^* is given according to (3.19) by

$$(4.15) \quad M(\xi^*, \lambda; \beta) = \sigma^2 + 2|\beta|^2 - 2|\beta^{(0)}|^2$$

$$= \sigma^2 + 2 \sum_{u=1}^{M-1} |\beta^{(u)}|^2$$

Thus, in case all $|\beta^{(u)}|^2$ have the same a-priori distribution, the risk under strategies (ξ^*, λ^-) and π will be

$$(4.16) \quad R(\xi^*, \lambda^*, \pi) = \sigma^2 + 2(M-1) R_{\pi}^*$$

Comparing (4.16) to (3.23) we conclude that Rao's g.l.s.e. $(L_V)^{-}$ might be very far from the optimum g.l.s.e. in case all the subvectors of β have approximately the same average effect. On the other hand, in case the effects of $\beta^{(1)}$ to $\beta^{(M-1)}$ are negligible relative to the effect of $\beta^{(0)}$ the Bayes g.l.s.e. will be very close to Rao's g.l.s.e.

Acknowledgment

The author gratefully acknowledges Professors H. Solomon and H. Chernoff for their helpful comments and encouragement.

References

1. A. P. Dempster (1960), "Random allocation designs I: On general classes of estimation methods", Ann. Math. Stat. Vol. 31, pp. 885-905.
2. A. P. Dempster (1961), "Random allocation designs II: Approximative theory for simple random allocations", Ann. Math. Stat. Vol. 32, pp. 387-405.
3. S. Ehrenfeld and S. Zacks (1961), "Randomization and factorial experiments", Ann. Math. Stat. Vol. 32, pp. 270-297.
4. S. Ehrenfeld and S. Zacks (1962), "Optimum strategies in factorial experiments", submitted for publication in the Ann. Math. Stat.
5. O. Kempthorne, The Design and Analysis of Experiments, John Wiley, New York (1952).
6. C.R. Rao (1962), "A note on a generalized inverse", Jour. Roy. Stat. Soc. (B) Vol. 24, p. 152.
7. B.V. Shah and O. Kempthorne (1962a), "Some properties of random allocation designs", Technical Report, Stat. Lab. Iowa Stat. Univ.
8. B.V. Shah and O. Kempthorne (1962b), "Randomization in fractional factorial", Technical Report, Stat. Lab. Iowa State Univ.
9. K. Takeuchi (1961), "On a special class of regression problems and its applications: Random Combined Fractional Factorial Designs", Rep. Stat. Appl. Res. JUSE, Vol. 7, No. 4, pp. 1-33.
10. K. Takeuchi (1961), "On a special class of regression problems and its application: some remarks about general models", Rep. Stat. Appl. Res. JUSE, Vol. 8, No. 1, pp. 7-17.
11. S. Zacks (1962), "On a complete class of linear unbiased estimators for randomized factorial experiments", submitted for publication in the Ann. Math. Stat.

STANFORD UNIVERSITY
TECHNICAL REPORTS DISTRIBUTION LIST
CONTRACT Nonw-225(52)

Armed Services Technical Information Agency Arlington Hall Station Arlington 12, Virginia	10	Commanding Officer Frankford Arsenal Library Branch, 0270, Bldg. 40 Bridge and Tacony Streets Philadelphia 37, Pennsylvania	1	Document Library U.S. Atomic Energy Commission 19th and Constitution Aves. N.W. Washington 25, D. C.	1
Bureau of Supplies and Accounts Code OW Department of the Navy Washington 25, D. C.	1	Commanding Officer Rock Island Arsenal Rock Island, Illinois	1	Headquarters Oklahoma City Air Materiel Area United States Air Force Tinker Air Force Base, Oklahoma	1
Head, Logistics and Mathematical Statistics Branch Office of Naval Research Code 436 Washington 25, D. C.	3	Commanding General Redstone Arsenal (ORDDIV-QC) Huntsville, Alabama	1	Institute of Statistics North Carolina State College of A & E Raleigh, North Carolina	1
Commanding Officer Office of Naval Research Branch Office Navy No. 100, Fleet P. O. New York, N. Y.	2	Commanding General White Sands Proving Ground (ORDBS-TS-TIB) Las Cruces, New Mexico	1	Jet Propulsion Laboratory California Institute of Technology Attn: A. J. Stosick 4800 Oak Grove Drive Pasadena 3, California	1
Commanding Officer Office of Naval Research Branch Office 1000 Geary Street San Francisco 9, California	1	Commanding General Attn: Paul C. Cox, Ord. Mission White Sands Proving Ground Las Cruces, New Mexico	1	Librarian The RAND Corporation 1700 Main Street Santa Monica, California	1
Commanding Officer Office of Naval Research Branch Office 10th Floor, The John Crerar Library Bldg. 86 East Randolph Street Chicago 1, Illinois	1	Commanding General Attn: Technical Documents Center Signal Corps Engineering Laboratory Fort Monmouth, New Jersey	1	Library Division Naval Missile Center Command U.S. Naval Missile Center Attn: J. L. Nickel Point Mugu, California	1
Commanding Officer Office of Naval Research Branch Office 346 Broadway New York 13, N. Y.	1	Commanding General Ordnance Weapons Command Attn: Research Branch Rock Island, Illinois	1	Mathematics Division Code 5077 U.S. Naval Ordnance Test Station China Lake, California	1
Commanding Officer Diamond Ordnance Fuze Labs. Washington 25, D. C.	1	Commanding General U.S. Army Electronic Proving Ground Fort Huachuca, Arizona Attn: Technical Library	1	NASA Attn: Mr. E. B. Jackson, Office of Aero Intelligence 1724 F Street, N. W. Washington 25, D. C.	1
Commanding Officer Picatinny Arsenal (ORDBB-TNB) Dover, New Jersey	1	Commander Wright Air Development Center Attn: ARL Tech. Library, WCRR Wright-Patterson Air Force Base, Ohio	1	National Applied Mathematics Labs. National Bureau of Standards Washington 25, D. C.	1
Commanding Officer Watertown Arsenal (OMRO) Watertown 72, Massachusetts	1	Commander Western Development Division, WDSIT P. O. Box 262 Inglewood, California	1	Naval Inspector of Ordnance U.S. Naval Gun Factory Washington 25, D. C. Attn: Mrs. C. D. Hock	1
Commanding Officer Attn: W. A. Labs Watertown Arsenal Watertown 72, Massachusetts	1	Chief, Research Division Office of Research & Development Office of Chief of Staff U. S. Army Washington 25, D. C.	1	Office, Asst. Chief of Staff, G-4 Research Branch, R & D Division Department of the Army Washington 25, D. C.	1
Commanding Officer Watervliet Arsenal Watervliet, New York	1	Chief, Computing Laboratory Ballistic Research Laboratory Aberdeen Proving Ground, Maryland	1	Superintendent U.S. Navy Postgraduate School Monterey, California Attn: Library	1
Commanding Officer Attn: Inspection Division Springfield Armory Springfield, Massachusetts	1	Director National Security Agency Attn: REMP-1 Fort George G. Meade, Maryland	2	Technical Information Officer Naval Research Laboratory Washington 25, D. C.	6
Commanding Officer Signal Corps Electronic Research Unit, EDL 9560 Technical Service Unit P. O. Box 205 Mountain View, California	1	Director of Operations Operations Analysis Div., AFOOP Hq., U.S. Air Force Washington 25, D. C.	1	Technical Information Service Attn: Reference Branch P. O. Box 62 Oak Ridge, Tennessee	1
Commanding Officer 9550 Technical Service Unit Army Liaison Group, Project Michigan Willow Run Research Center Ypsilanti, Michigan	1	Director Snow, Ice & Permafrost Research Establishment Corps of Engineers 1215 Washington Avenue Wilmette, Illinois	1	Technical Library Branch Code 234 U.S. Naval Ordnance Laboratory Attn: Clayborn Graves Corona, California	1
Commanding Officer Engineering Research & Development Labs. Fort Belvoir, Virginia	1	Director Lincoln Laboratory Lexington, Massachusetts	1	Institute for Defense Analyses Communications Research Division von Neumann Hall Princeton, New Jersey	1
		Department of Mathematics Michigan State University East Lansing, Michigan	1		

August, 1962

Mr. Irving B. Altman Inspection & QC Division Office, Asst. Secretary of Defense Room 2B870, The Pentagon Washington 25, D. C.	1	Professor Solomon Kullback Department of Statistics George Washington University Washington 7, D. C.	1	Professor L. J. Savage Mathematics Department University of Michigan Ann Arbor, Michigan	1
Professor T. W. Anderson Department of Statistics Columbia University New York 27, New York	1	Professor W. H. Kruskal Department of Statistics The University of Chicago Chicago, Illinois	1	Professor W. L. Smith Statistics Department University of North Carolina Chapel Hill, North Carolina	1
Professor Robert Bechhofer Dept. of Industrial and Engineering Administration Sibley School of Mechanical Engineering Cornell University Ithaca, New York	1	Professor Eugene Lukacs Department of Mathematics Catholic University Washington 15, D. C.	1	Dr. Milton Sobel Statistics Department University of Minnesota Minneapolis, Minnesota	1
Professor Fred. C. Andrews Department of Mathematics University of Oregon Eugene, Oregon	1	Dr. Craig Maguire 2954 Winchester Way Rancho Cordova, California	1	Mr. G. P. Stock Division 5511 Sandia Corp., Sandia Base Albuquerque, New Mexico	1
Professor Z. W. Birnbaum Department of Mathematics University of Washington Seattle 5, Washington	1	Professor G. W. McElrath Department of Mechanical Engineering University of Minnesota Minneapolis 14, Minnesota	1	Professor Donald Truxx Department of Mathematics University of Oregon Eugene, Oregon	1
Dr. David Blackwell Department of Mathematical Sciences University of California Berkeley 4, California	1	Dr. Knox T. Millsaps Executive Director Air Force Office of Scientific Research Washington 25, D. C.	1	Professor John W. Tukey Department of Mathematics Princeton University Princeton, New Jersey	1
Professor Ralph A. Bradley Department of Statistics Florida State University Tallahassee, Florida	1	D. E. Newnam Chief, Ind. Engr. Div. Comptroller Hq., San Bernardino Air Materiel Area USAF, Norton Air Force Base, California	1	Professor G. S. Watson Department of Mathematics University of Toronto Toronto 5, Ontario, Canada	1
Dr. John W. Cell Department of Mathematics North Carolina State College Raleigh, North Carolina	1	Professor Edwin G. Olds Department of Mathematics College of Engineering and Sciences Carnegie Institute of Technology Pittsburgh 13, Pennsylvania	1	Dr. Harry Weinberger Special Projects Office, SP2016 Navy Department Washington 25, D. C.	1
Professor William G. Cochran Department of Statistics Harvard University 2 Divinity Avenue, Room 311 Cambridge 38, Massachusetts	1	Dr. William R. Pabst Bureau of Weapons Room 0306, Main Navy Department of the Navy Washington 25, D. C.	1	Dr. F. J. Weyl, Director Mathematical Sciences Division Office of Naval Research Washington 25, D. C.	1
Miss Besse B. Day Bureau of Ships, Code 3420 Room 3210 Main Navy Department of the Navy Washington 25, D. C.	1	Mr. Edward Paulson 72-10 41 Ave. Woodside 77 New York, New York	1	Dr. John Wilkes Office of Naval Research, Code 200 Washington 25, D. C.	1
Dr. Walter L. Deemer, Jr. Operations Analysis Div., DCE/O Hq., U.S. Air Force Washington 25, D. C.	1	H. Walter Price, Chief Reliability Branch, 750 Diamond Ordnance Fuze Laboratory Room 105, Building 83 Washington 25, D. C.	1	Professor S. S. Wilks Department of Mathematics Princeton University Princeton, New Jersey	1
Professor Cyrus Derman Dept. of Industrial Engineering Columbia University New York 27, New York	1	Professor Ronald Pyke Mathematics Department University of Washington Seattle 5, Washington	1	Mr. Silas Williams Standards Branch, Proc. Div. Office, DC/S for Logistics Department of the Army Washington 25, D. C.	1
Dr. Donald P. Gaver Westinghouse Research Labs. Beulah Rd. - Churchill Boro. Pittsburgh 35, Pa.	1	Dr. Paul Rider Wright Air Development Center, WCCRM Wright-Patterson A.F.B., Ohio	1	Professor Jacob Wolfowitz Department of Mathematics Cornell University Ithaca, New York	1
Mr. Harold Gumbel Head, Operations Research Group Code 01-2 Pacific Missile Range Box 1 Point Mugu, California	1	Professor Herbert Robbins Dept. of Mathematical Statistics Columbia University New York 27, New York	1	Mr. William W. Wolman Code MER - Bldg. T-2 Room C301 700 Jackson Place, N. W. Washington 25, D. C.	1
Dr. Ivan Hershner Office, Chief of Research & Dev. U. S. Army, Research Division 3E382 Washington 25, D. C.	1	Professor Murray Rosenblatt Department of Mathematics Brown University Providence 12, Rhode Island	1	Marvin Zelen Mathematics Research Center U. S. Army University of Wisconsin Madison 6, Wisconsin	1
Professor W. Hirsch Institute of Mathematical Sciences New York University New York 3, New York	1	Professor Herman Rubin Department of Statistics Michigan State University East Lansing, Michigan	1	Additional copies for project leader and assistants and reserve for future requirements	50
Mr. Eugene Hixson Code 600.1 GSFC, NASA Greenbelt, Maryland	1	Professor J. S. Rustagi College of Medicine University of Cincinnati Cincinnati, Ohio	1		
Professor Harold Hotelling Department of Statistics University of North Carolina Chapel Hill, North Carolina	1	Professor I. R. Savage School of Business Administration University of Minnesota Minneapolis, Minnesota	1		
		Miss Marlon M. Sandomire 2281 Cedar Street Berkeley 9, California	1		

Contract Nonr-225(52)
August, 1962

JOINT SERVICES ADVISORY GROUP

Mr. Fred Frishman Army Research Office Arlington Hall Station Arlington, Virginia	1	Lt. Col. John W. Querry, Chief Applied Mathematics Division Air Force Office of Scientific Research Washington 25, D. C.	1
Mrs. Dorothy M. Gilford Mathematical Sciences Division Office of Naval Research Washington 25, D. C.	3	Major Oliver A. Shaw, Jr. Mathematics Division Air Force Office of Scientific Research Washington 25, D. C.	2
Dr. Robert Lundegard Logistics and Mathematical Statistics Branch Office of Naval Research Washington 25, D. C.	1	Mr. Carl L. Schaniel Code 122 U.S. Naval Ordnance Test Station China Lake, California	1
Mr. R. H. Noyes Inst. for Exploratory Research USASRDL Fort Monmouth, New Jersey	1	Mr. J. Weinstein Institute for Exploratory Research USASRDL Fort Monmouth, New Jersey	1